International Journal of Multidisciplinary Trends

E-ISSN: 2709-9369
P-ISSN: 2709-9350
www.multisubjectjournal.com
IJMT 2025; 7(1): 72-74
Received: 12-10-2024
Accepted: 15-11-2024

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Injective projective and continuous modules and their relationships to some rings: A critical review

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DOI: https://dx.doi.org/10.22271/multi.2025.v7.i1b.572

Abstract
Projective and Injective modules arise quite abundantly in nature. For example, all free modules that we know of, are projective modules. Similarly, the group of all rational numbers and any vector space over any field are examples of injective modules. In this paper, we study the theory of projective and injective modules.
This expository note delves into the theory of projective modules parallel to the one developed for injective modules by Matlis. Given a perfect ring R, we present a characterization of indecomposable projective R-modules and describe a one-to-one correspondence between the projective indecomposable R-modules and the simple R-modules.

Keywords: Projective, r-modules, injective modules, rational numbers

Introduction
Throughout this note, R denotes an associative ring with identity. All modules are assumed to be left and unitary.
There is a well-known theorem in the representation theory of finite-dimensional algebras over a field which provides a 1-1 correspondence between the isomorphism classes of indecomposable projective modules over the algebra on the one hand, and the isomorphism classes of simple modules over the algebra on the other hand. More specifically, let k be a field and a finite-dimensional k-algebra. Then every simple A-module is a quotient of some indecomposable projective A-module which is unique up to isomorphism. Conversely, for every indecomposable projective A-module P, there is a simple A-module, unique up to isomorphism that is a quotient of P by some maximal submodule; see [Le]. For a generalization of this result to Artin algebras, see, and to perfect rings. The purpose of this note is to present a novel descriptive proof of this theorem in the following more comprehensive form.

Proposition 1: Let R be a perfect ring, and P a nonzero R-module. Then the following assertions are equivalent:
1. P is an indecomposable projective R-module.
2. P is a sum-irreducible projective R-module.
3. P is the projective cover of its every nonzero quotient module.
4. P is the projective cover of R/m for some maximal left ideal m of R. Further if R is commutative, then the above assertions are equivalent to the following one:
5. $P \cong R_m$ for some maximal ideal m of R.

Proposition 2: Let R be a perfect ring. There is a one-to-one correspondence between the indecomposable projective R-modules and the simple R-modules.

A Decomposition theorem for injective modules
We recall that a module is named directly *finite* if it is not isomorphic to a proper direct summand of itself, and *purely infinite* if it is isomorphic to the direct sum of two copies of itself.
We remind the reader that a directly finite injective module has the cancellation property. An injective module is not directly finite, if and only if it contains an infinite direct sum of nonzero pairwise isomorphic submodules, if and only if it has a nonzero purely infinite direct summand. With these concepts, we shall prove the following decomposition theorem for arbitrary injective modules.

Theorem 1: Every injective module E has a direct decomposition, E = U⊕ V, where U is directly finite, V is purely infinite, and U and V have no nonzero isomorphic summands (or submodules). If $E = U_1 \oplus V_1 = U_2 \oplus V_2$ are two such decompositions, then $E = U_1 \oplus V_2$ holds too, and consequently $U_1 \cong U_2$ and $V_1 \cong V_2$ We begin with an auxiliary observation.

Lemma 2: Let A be a submodule of a module C, let EA be directly finite, and let C be subisomorphic to an injectivt' module 1. Then every monomorphism f: A → I extends to a monomorphism C → 1.

Proof: The given monomorphisms f: A → I and g: C → I exten to monomorphisms ϕ:EA → I and γ: EC → I. We have EA⊕X=EC, hence I=γ(EC)⊕Y=v(EA)⊕γ(X)⊕ Y, as well as I=c∅(EA)⊕Z.
As ∅(EA) ≅EA ≅γ(EA) is directly finite and injective, hence has the cancellation property, we conclude Z≅γ(X)⊕ Y≅ X⊕Y. We obtain a monomorphism, μ: X → Z.
Then $\varphi \oplus \mu : EC = EA \oplus X \to \varphi(EA) \oplus Z = I$ is a monomorphism, bv whose restriction $\varphi \oplus \mu | C$ extends, f.

Injective and projective modules and h-ring

In this section, we studied several relationships between injective, projective modules and H-ring.
(*) "Every injective R-module is a lifting. Also, if R is Artinian ring and every f.g. injective R-module is lifting, then R is H-ring".

Theorem 2: Let M be an R-module. If M is a regular, then R is H-ring.

Proof: Firstly, we know that M is regular module iff for all m ∈M, f(m)∈Hom_R (M, R), then mf(m)=m. Or, iff for all m ∈M, ∃ N ≤M ∈M=m⊕N. If f(m)=e, then mf(m)=me=m. So, f(rm)=rf(m)=re and hence $r_1e=r_2e$ ($r_1m=r_2m$). Therefore, $r_1mf(m)=r_2mf(m)$ and then $r_1em=r_2em$ (one to one). Thus, M is injective and by (*) R is H-ring.

Theorem 3: Let R be a perfect regular ring. If M is a f. generated module, then R is H-ring.

Proof: Since R is regular ring, then M is flat. But R is perfect ring. So M is projective with f.g. property we obtain M=⊕Σei. Thus, M is injective and so lifting. Thus, R is H-ring. Now we study the relationship between multiplication faithful module and H-ring.

Theorem 4: Let M be an R-module. If M is a multiplication and faithful R-module, then R is H-ring.

Proof: Suppose that M is a multiplication R-module. We should prove that M is a torsion free. Take M is not torsion free. Then M is a torsion (∃m∈M is a torsion element, ∃c∈R and 0≠m∈M∋cm=0) such that c one of the regular element in R (non-zero divisors). There exists an ideal I ∈ R ∋Rm=Im, because M is a multiplication module. Then cI≠0, I≠0. So cIM=crm and hence cIM=0 (cI≠0). So M is unfaithful module and this contradiction. Therefore, M is a torsion free (M≈I). Hence there exists a homomorphism is one to one (M is injective module). Thus, R is H-ring.

Lemma 1. Let M be a module over P.I.D. So M is injective iff it is divisible**.**

Proof ⟹
Let A be injective. Not that Rr is an ideal of R. We define f:Rr→A by f(br)=ba and f(cr)=ba, a∈A. Now every two elements b, c ∈R, we get
f(br+cr)=f((b+c)r)
=(b+c)a
=ba+ca
=f(br)+f(cr)
and
f(b(cr))=f((bc)r)
=(bc)a
=b(ca)
=bf(cr).

Hence f is R-homomorphism. M is injective, so ∃ g: R→A ∋ g| Rr = f.
Then
a=1.a
=f(1.r)
=g(1.r)
=rg(1) ∃g(1) ∈A.
Thus, A is a divisible module.
⟸ Suppose M is a divisible and f:I→M is an R-homo. and I is an ideal. We have R is P.I.D., then I=Rb. Since M is a divisible, then ∃a∈M∋f(b)=ba. Define g(r)=ra ∃g is an R-homo. Every bc ∈R, then g(bc)=(st)a=c(ba)=cf(b)=f(cb). Thus, M is injective module.

Theorem 5: If R be a Euclidean domain and M is a divisible over R, then R is H-ring.

Theorem 6: Suppose M is divisible over Dedekind domain R with f. g. prime ideals. So R is H-ring.

Corollary 1: Let R be a P.I.D. ring. If M is a divisible R-module, then R is H-ring

Example 1: Q is an injective Z-module, so Z is H-ring.

Example 2: Z is not injective Z-module, so Z is not H-ring.

Corollary 2: The following are equivalent:
1. R is regular ring.
2. Every cyclic R-module is a flat.
3. Every simple R-module M is injective and R is H-ring.

Theorem 7: Suppose that M is a free R-module over R with C_1 and C_2 properties, then R is QF-ring and so is CO-H-ring.

Proof: We have M is a free. So M is a projective. But M satisfies C_1 and C_2, then it is continuous module. So, R is a QF-ring and then CO-H-ring.

Theorem 8: Let R be a ring. If M is a injective R-module with C_1 and C_3 properties, then R is QF-ring and so is COH-ring.
Proof: Same proof Theorem. If M satisfies C_1 and C_3, then it is semi-continuous.

Corollary 3: Let R be a P.I.D. ring. If M is a divisible R-module with D_1 and D_2 properties, then R is QF-ring and so is CO-H-ring.

Proof: Since R is a P.I.D. ring and M is a divisible module, then M is injective. So M is projective. We have M satisfies D_1 and D_2 properties imply M is a semi-perfect module. So R is a QF-ring. Thus, R is a CO-H-ring.

Corollary 4: If M is a regular module over P.I.D with D_1 and D_3 properties, then R is QF-ring and so is CO-H-ring.

Proof: Since M is a regular module, then it is injective and hence it is a projective module. Also, from D_1 and D_3 properties, we obtain quasi-semi-perfect. So, R is a QF-ring. Thus, R is a CO-H-ring. Let R be a Dedekind domain and a nonzero ideal I_1 be a (FI) of R. If M is a divisible and I2 is an integral ideal, then M is injective (R CO-H-ring).

Theorem 9: Let R be a Dedekind domain with finitely generated prime ideals. If M is a divisible, then it is injective. So, R is a QF-ring (R is CO-H-ring).

Corollary 5: A divisible module M over Euclidean domain R is injective and hence R is CO-H-ring.

Proof: Let $I_1 \lhd R$. So $I_1=\{0\}=()$ or let $0\neq \alpha$ in I_1 $\exists d(\alpha)$ is a least, so any β in I_1 we get $\beta=q\,\alpha+r$ and $r=0$ or $d(r)<d(\alpha)$ But $r=q-\beta$ in I_1. Since $d(\alpha)$ is a minimal and $r=0$, then $\alpha|\beta$ and $I_1=(\alpha)$. Hence R is a principal ideal domain with M divisible module, we obtain M is an injective module and so M is quasi-semi perfect. Thus, R is a QF-ring and hence is CO-H-ring.

Corollary 6: Let R be a ring. If M is a multiplication and torsion free R-module, then there exists an ideal I of R s.t M≈I (∃ f homomorphism is one to one). So M is injective module and then R is H-ring (R is a CO-H-ring).

Applications

We shall see how the preceding results can be used to extend, with little effort, many theorems from injective to (quasi-)continuous modules.

Thus, the above Theorem holds for quasi-continuous modules. Moreover, if V' is any purely infinite submodule of a quasi-continuous module Al, then there exists a decomposition M= U⊕ V such that V' ⊂ V.

Proof. We note first that if M= A ⊕B is any decomposition of a quasi-continuous module M, such that A and B have no nonzero isomorphic summands, then they have no nonzero isomorphic submodules either. Indeed, if X and Y are isomorphic submodules of A and B respectively, then the quasi-continuity implies that X and Y are essential in summands P and Q of A and B, hence of M.

Conclusion

Injective and projective modules are two of important concepts in module theory. In this paper we have defined injective module as an algebraic structure. Some basic properties have been introduced. It has been shown that if R is a ring and M is a free R-module with C_1 and C_2 properties, then R is QF-ring and so is CO-H-ring. The main result is if R is a Dedekind domain with finitely generated prime ideals and M is a divisible R-module, then it is an injective with R is a QF-ring, so R is CO-H-ring. Also, we proved that if R is a ring and M is a multiplication with faithful R module, then R is H-ring. Finally, we investigated many results about H-ring and CO-H-ring by using others modules such as divisible and regular modules.

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