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Introducing Bessel function and their properties

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Abstract
A hybrid approach to the introduction of Bessel function is proposed. Combining the factorization method for resolving second-order homogeneous differential equation into a ladder-operator representation with the Laplace transform method for solving the zero-order Bessel equation, the standard infinite series solutions to the Bessel equation are determined, as well as many well-known relation involving Bessel functions.
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1. Introduction

The importance of the Bessel equation (in its various form) and the resulting Bessel function [3, 8, 10, 12, 14, 17, 20] and it follows, naturally, that an introduction to Bessel function $j_n(x)$ trough a hybrid approach to the solution of the Bessel equation of integral-order, that is with n (at first) a non-negative integer. this hybrid approach combines the determination of 'raising' and 'lowering' differential operators (from the factorization of Bessel's equation, recognizable as standard recurrence relation for Bessel functions [11]) with the solution of the zero-order Bessel equation ($n = 0$) through the application of the Laplace transform [5, 15, 18]. The solution of the zero-order Bessel equation via the Laplace transform can be expressed in infinite series form(through a judicial application of the Binomial theorem; see below) and then, using, recursively, the 'raising' ladder-operator (and standard results on the manipulation of infinite series) the Bessel function solution for integral-order Bessel equation (1.1) can be determined (through mathematical induction. essentially). Along the way, basic results in the manipulation of infinite series are quoted (sourced) as required. Note that we are only after particular solutions of (1.1) and we do not consider the topic of the second solution here.

$$x^2\frac{d^2j_n(x)}{dx^2}+x\frac{dj_n(x)}{dx}+(x^2-n^2)j_n(x)=0\ldots\ldots\ldots\ldots\ldots\ldots(1.1)$$

The paper is organized as follows. In the next section, section2 we consider the Bessel equation of integral-order and show, by operator factorization, how the Bessel equation may be factorized in term of 'raising' and 'lowering' ladder-operators. The 'raising' and 'lowering' ladder-operators enable us to relate Bessel functions of different integral-order, provided we are able to start-off the 'raising' process with a knowledge of the Bessel function of zero-order. To find the zero-order Bessel function apply, in section 3, the Laplace transform to the zero-order Bessel equation and express the solution of the transformed equation in infinite series form in the transform variable using the binomial theorem; on inverting the infinite series in the transform variable term-by-term we obtain the infinite series form of the solution to the zero-order Bessel equation, that is, the zero-order Bessel function. It is now left for us to obtain the form of the integral order Bessel function, from the zero-order Bessel function, using the 'raising' ladder-operator. This is done in section 4, where we derive the Rayleigh formula form of the Bessel function of integral-order which leads from the infinite series form of the solution to the zero-order Bessel function to the infinite series form of the solution to the general integral-order Bessel function. The paper rounds-off with a discussion and conclusions section, section 5.

2. Factorizing the Bessel Equation-Ladder-Operators

The frost point about factorizations of (1.1) is that they are not of the classic form.

$$[x^2\frac{d^2x}{dx^2}+x\frac{d}{dx}+(x^2-n^2)]j_n(x)$$

$$\equiv [p_2(x)\frac{d}{dx}+q_2(x)][p_2(x)\frac{d}{dx}+q_2(x)]j_n(x)= 0 \ldots\ldots (2.1)$$

With the $p's$ and $q's$ to be discovered. In fact, such a factorization (2.1) 0f (1.1) cannot be determined, as was shown by Guerra and Shapiro [10] (see, also, [2].
Instead, we have to search for factorizations of the form

$$[\frac{d^2x}{dx^2}+\frac{1}{x}\frac{d}{dx}+(1-\frac{n^2}{x^2})]j_n(x)=[(\frac{d}{dx}-p_1(x))(\frac{1}{x}\frac{d}{dx}-p_2(x))+K]j_n(x)=0\ldots\ldots(2.2)$$

Which is reminiscent of the treatment of the Harmonic oscillator problem in quantum mechanics [9]. By comparing the left-hand- side (2.2) with expanded middle of (2.2) we hope to determine the unknown constant K. So, expanding the middle of (2,2) we get the identity

$$[\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+(1-\frac{n^2}{x^2}]j_n(x)=[\frac{d^2}{dx^2}-(p_1+p_2)\frac{d}{dx}+(K+p_1p_2-p_{2\prime})]j_n(x)\ldots\ldots(2.3)$$

From which it follows, by obvious comparison, that we must try to determine the *three* unknown $p_1(x), p_2(x)$ and k from the *tow* equation

$$(p_1+p_2)=-\frac{1}{x}\ldots\ldots\ldots(2,4a)$$

And

$$K+p_1p_2+p_{2\prime}=1-\frac{n^2}{x^2}\ldots\ldots\ldots\ldots(2.4b)$$

Eliminating $p_1(x)$ from (2,4b), we find that the problem resolves to find a particular solution to Riccati equation.

$$p_{2\prime}+\frac{1}{x}p_2+p_2^2-(k-1)-\frac{n^2}{x^2}=0\ldots\ldots\ldots\ldots\ldots(2.5)$$

There is no general means of solving the Riccati equation [11].
Therefore, as a *trial solution* to (2.5) we set $k=1$ and $p_2=\frac{c}{x}$, with the constant c to be determined. On substituting these assumptions into (2.5) we find that we require that $c^2=n^2$ or $c=+n,-n$ so that on substituting back we get.

$$p_2(x)=+\frac{n}{x},\frac{n}{x} \text{ and } p_1(x)=\frac{-n-1}{x},\frac{+n-1}{x}\ldots\ldots\ldots\ldots\ldots\ldots(2.6)$$

Choosing the plus sign for $p_1(x)$ in (2.6), and therefore the minus sign for $p_2(x)$, we find then that the Bessel equation (1.1) factorizes, via (2.2), as

$$[\left(\frac{d}{dx}-\frac{n-1}{x}\right)\left(\frac{d}{dx}+\frac{n}{x}\right)+1]j_n(x)=0\ldots\ldots\ldots\ldots\ldots(2.7a)$$

While choosing the minus sign for $p_1(x)$ in (2.6), and therefore the plus sign for $p_2(x)$, yields another factorization of the Bessel equation (1.1), via (2.2), as

$$[\left(\frac{d}{dx}+\frac{n+1}{x}\right)\left(\frac{d}{dx}-\frac{n}{x}\right)+1]j_n(x)=0\ldots\ldots\ldots\ldots(2.7b)$$

Check ink the equivalence of equation (2.7) and (1.1) is then an elementary exercise in differentiation.
Introduce the ladder-operators associated with (1.1), we write-out equation (2.7) in two, equivalent, operator-factorized form as (see, also chin and Barrett [4] on this point)

$$\left(\frac{d}{dx}-\frac{(n-1)}{x}\right)\left(\frac{d}{dx}+\frac{n}{x}\right)j_n(x)=-j_n(x)\ldots\ldots\ldots\ldots.(2.8a)$$

or

$$\left(\frac{d}{dx}+\frac{(n+1)}{x}\right)\left(\frac{d}{dx}-\frac{n}{x}\right)j_n(x)=-j_n(x)\ldots\ldots\ldots\ldots\ldots.(2.8b)$$

Examination of equation (2.8) shows that we may 'decouple' equation (2.8) into two operator relations between successive Bessel functions that are again equivalent to (1.1) and, of course (2.8): this involve the 'raising' and 'lowering' operators mentioned in the introduction and are written in operator format as

$$\left(\frac{d}{dx}+\frac{n}{x}\right)j_n(x)=j_{n-1}(x)\ldots\ldots\ldots.(2.9a)$$

and

$$\left(\frac{d}{dx}-\frac{n}{x}\right)j_n(x)=-j_{n+1}(x)\ldots\ldots\ldots\ldots.(2.9b)$$

Apparently, $\left(\frac{d}{dx}+\frac{n}{x}\right)$ is the 'lowering' operator while $\left(\frac{d}{dx}-\frac{n}{x}\right)$ is the 'raising' operator for $j_n(x)$ [2, 4]. So, given $j_0(x)$, we may use (2.9b) to get $j_n(x)$ for any integer $n \geq 1$. Note that we may also write our 'raising' and 'lowering' operations using the other differential operators in (2.8), that is we have.

$$\left(\frac{d}{dx}+\frac{n+1}{x}\right)j_{n+1}(x)=j_n(x)\ldots\ldots\ldots\ldots.(2.10a)$$

and

$$\left(\frac{d}{dx}-\frac{(n-1)}{x}\right)j_{n-1}(x)=-j_n(x)\ldots\ldots\ldots\ldots.(2.10b)$$

Of course, the relation (2.9) and (2.10) are well-known as differential recurrence relations for the integral-ordered Bessel function [1, 4, 15].
Further, if we eliminate the derivative terms from equations (2.9), we get (as a 'bonus') the three-term recurrence relation for $j_n(x)$, that is [1.15].

$$2nj_n(x)-xj_{n-1}(x)-xjn+1(x)=0\ldots\ldots\ldots\ldots\ldots.(2.11)$$

We must determine, now, the solution of the Bessel equation of zero-order, which is required to 'start-off' the ladder-operator determination of higher-order Bessel functions; to this end, we employ the Laplace transform.

3. Laplace Transform Solution of Zero-Order Bessel Equation

Consider, now, the Bessel equation of zero-order, that is

$$x\frac{d^2 j_n(x)}{dx^2}+\frac{d j_n(x)}{dx}+xj_n(x)=0\ldots\ldots\ldots\ldots\ldots.(3.1)$$

We will solve (3.1) using the Laplace transform (the method is well-known and well-understood [5.18] and rewrite the solution, using the Binomial theorem, as an infinite series in the transformed variable; then, we will invert the infinite series in the transformed variable (term-by-term) to obtain the usual infinite series solution to (1.1) in terms of the original Variable x.
First, we introduce the Laplace transform $L[f(x)] \equiv F(s)$ of the function $f(x)$ as the usual improper integral [15, 16].

$$L[f(x)] \equiv F(s)=\int_0^\infty e^{-sx}f(x)dx\ldots\ldots\ldots\ldots\ldots.(3.2)$$

And where we will make use of the following two well-known [6] properties of the Laplace transform (with zero initial condition, as we seek *particular solution*)

$$L[f^{(k)}(x) \equiv s^k F(s)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.3a)$$

and

$$L[x^k f(x)] \equiv (-1)^k (L[f(X)])^{(k)}\ldots\ldots\ldots\ldots\ldots\ldots(3.3b)$$

Where the superscripts imply differentiation with respect to either x or s.
From equation (3.3), we have the following result [6, 18] helping us to transform (3.1):

$$L[xf^{(2)}(x)] \equiv (-1)^1 (L[f^{(2)}(x)])^{(1)} = (s^2F(s))^{(1)} = -s^2F'(s) - 2sF(s)\ldots\ldots\ldots\ldots\ldots\ldots(3.4)$$

next, (see, also, chorlton [4]) we Laplace transform (3.1), term-by-term using (3.3) and (3.4), to get a differential equation for the transform $j_0(s)$ of $j_0(x)$:

$$s^2 + 1)j'_0(s) + sj_0(s) = 0\ldots\ldots\ldots\ldots\ldots\ldots(3.5)$$

(Returning to the 'dash' notation for differentiation) with the *variable separable* equation (3.5) having the particular solution

$$j_0(s) = (s^2+1)^{\frac{-1}{2}}\ldots\ldots\ldots\ldots\ldots\ldots(3.6)$$

Now, to get a series solution to our transformed problem, we expand (3.6) via the Binomial theorem to get

$$j_0(s) = s^{-1}(1+s^{-2})^{\frac{-1}{2}} = \sum_{k=0}^{\infty} (k^{\frac{-1}{2}})s^{-(2k+1)}\ldots\ldots\ldots\ldots\ldots\ldots(3.7)$$

Which we may invert term-by-term [6] using the result

$$L[x^k] = \int_0^{\infty} e^{-sk}x^k dx = \frac{k!}{s^{K+1}}\ldots\ldots\ldots\ldots\ldots\ldots(3.8)$$

To get the infinite series expansion for $j_0(s)$ in the form

$$j_0(x) = \sum_{k=0}^{\infty} (k^{\frac{-1}{2}})\frac{x^{2k}}{(2k)!}\ldots\ldots\ldots\ldots\ldots\ldots(3.9)$$

Finally, to get the infinite series expansion for $j_0(s)$ in the *standard* form, we evaluate the generalized Binomial coefficient to find that

$$(k^{\frac{-1}{2}}) = \frac{(\frac{-1}{2})(\frac{-1}{2}-1)(\frac{-1}{2}-2)\ldots(-\frac{1}{2}-(k-1))}{k!} = \frac{(-1)^k}{2^k k!}(2k-1)!!\ldots\ldots\ldots\ldots\ldots\ldots(3.10)$$

With

$$(2k-1)!! = (2k-1)(2k-3)(2k-5)\ldots5.3.1 = \frac{(2k)!}{2^k k!}\ldots\ldots\ldots\ldots\ldots\ldots(3.11)$$

Where we have induced in (3.11) the *double* factorial function $(2k-1)!!$ along with its main identity [7.18]. So, substituting (3.11) into (3.10) and then (3.10) into (3.9) and re-arranging, we get the *standard* form of the infinite series expansion for $j_0(s)$ as [5, 13, 17, 19]

$$j_0(x) = \sum_0^{\infty} \frac{(-1)^k}{(k!)^2}(\frac{x}{2})^{2k}\ldots\ldots\ldots\ldots\ldots\ldots(3.12)$$

4. Solution of the Integral-Order Bessel function

To produce a series solution for the general integral-order Bessel equation (1.1), we make repeated use of the raising-operator (4.1) below:

$$(\frac{d}{dx}-\frac{n}{x})j_n(x) = -j_{n+1}(x)\ldots\ldots\ldots\ldots\ldots\ldots(4.1)$$

Introducing the 'integrating factor'

$$\exp(-\int \frac{n}{x}dx) = x^{-n}\ldots\ldots\ldots\ldots\ldots\ldots(4.2)$$

We 'solve' (4.1) to get the equivalent relation

$$(\frac{1}{x}\frac{d}{dx})[\frac{j_n(x)}{x^n}] = -\frac{j_{n+1}(x)}{x^{n+1}}\ldots\ldots\ldots\ldots\ldots\ldots(4.3)$$

Repeated application of the operator $\frac{1}{x}\frac{d}{dx}$ leads to the recursive formula

$$(\frac{1}{x}\frac{d}{dx})^m[\frac{j_n(x)}{x^n}] = (-1)^m\frac{j_{n+m}(x)}{x^{n+m}}\ldots\ldots\ldots\ldots\ldots\ldots(4.4)$$

For integer $m \geq 1$. The relation (4.4), if necessary, may be bolstered by an induction argument. Cross-multiplying, we find that (4.4) is just another way of writing the Rayleigh formula [5, 13], that is

$$j_{n+m}(x) = (-1)^m x^{n+m}(\frac{1}{x}\frac{d}{dx})^m[\frac{j_n(x)}{x^n}]\ldots\ldots\ldots\ldots\ldots\ldots(4.5)$$

so that, with $n = 0$, (4.5) enables us to obtain $j_m(x), m \geq 1$, from $j_0(x)$ as [1]

$$j_m(x) = (-1)^m x^m(\frac{1}{x}\frac{d}{dx})^m[j_0(x)]\ldots\ldots\ldots\ldots\ldots\ldots(4.6)$$

To get the series form of $j_n(x)$, from (4.6) and (3.12), we change the dummy index in (4.6) and write

$$j_n(x) = (-1)^n x^n(\frac{1}{x}\frac{d}{dx})^n[\sum_{k=0}^{\infty}\frac{(-1)^K}{(k!)^2}(\frac{x}{2})^{2k}]\ldots\ldots\ldots\ldots\ldots\ldots(4.7)$$

To perform the derivatives in (4.6), the chain rule for derivatives is required (see Charlton [5]). On making the substitution $u = x^2$, we may rewrite (4.6) as

$$j_n(x) = (-1)^n 2^n x^n \sum_{k=n}^{\infty}\frac{(-1)^k}{2^{2k}\,k!(k-n)!}u^{k-n}\ldots\ldots\ldots\ldots\ldots\ldots(4.8)$$

Then, on substituting back $u = x^2$, multiplying through and changing the dummy index again, (4.7) becomes the usual series representation of the general-order Bessel function [5, 13, 17, 19]:

$$j_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!(r+n)!}(\frac{x}{2})^{n+2r}, n = 0,1,2,3,\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(4.9)$$

Finally, as well known [13], for *negative* integers $n = -1, -2, -3, \ldots$

$$j_n(x) = (-1)^n j_n(x)\ldots\ldots\ldots\ldots\ldots\ldots(4.10)$$

Which follows form (4.8) by replacing n by $(-n)$, setting negative factorial equal to zero and re-setting the dummy variable.

We have at last achieved our objective in (4.9) (and (4.10), where we have a particular solution of (1.1) in terms of the infinite series (4.9), as well as in the three main recurrence relation for the integral-order Bessel functions, that is, equation (2.9)/(2.10), equation (2.11) and the Rayleigh formula (4.5)/(4.6).

5. Discussion and conclusions

On consideration, it is remarkable just how many topics we have touched on and it is interesting to look back over the paper and list these topics and consider their appearance and interaction. To begin with, we have had to deal with the Bessel equation itself (1.1), a Cacti equation (2.5), a variable-separable equation (3.5) and a differential expression (4.1) requiring an integrating factor. We have occupied ourselves, also, with various general techniques in the pursuit of our goal: a factorization approach for second-order linear differential operators-leading to a ladder-operator representation of the determination of the general-order Bessel function - then the Laplace transform solution of a second-order linear ordinary differential equation with *variable* coefficients and then, to crown this off as it were, the introduction and manipulation of the infinite series representation of the zero-order and then the general-order Bessel function through the manipulation of the Binomial theorem. We now discuss these points in more detail, meanwhile drawing attention to certain other factors that must be considered when presenting the material presented and discussed above.

First, to obtain the factorizations (2.8) of (1.1) it was necessary to make an assumption about the basic form of the factorized equation, that is, equation (2.2) as mentioned at the time, the more usual form of factorization (equation (2.1) is not possible for the Bessel equation (1.1) [10]; however, another type of factorized representation for a second-order differential operator was available as an exemplar in the case of the factorization of the quantum simple harmonic oscillator equation [9] and we have (essentially) followed this approach. critical to this analysis was the generation and (particular) solution of a Riccati equation, equation (2.5). As there is no general technique for attacking such problems [11] was necessary to *assume* both a value for λparticular form for $p_2(X)$; the particular choice of λ was also forced on us by the fact that the factorization (2.2) contains three unknowns, but only two relation between the unknowns emerge from the identity (2.2)/(2.3). (Given a particular solution to a Riccati equation though, the general solution can be constructed in a well-known manner [11] of course, it was through an examination of the factorization (2.8) that we were able to extract the raising and lowering operators in (2.9) [4], which also led to the three-term recurrence relation (2.1) [14].

Next, turning our attention back to the Laplace transform solution of (3.1), first we encounter a standard variable separable differential equation in a novel situation and then we realize that certain conditions must be satisfied and, in a class room, discussed, for the term-by-term manipulation of the series (3.7) and (3.12) to be valid. The Binomial expansion and its convergence are well understood [5, 15], while the term-by-term inversion of (3.7) can be treated as an *operational* procedure based on (3.8) [6]. The resulting infinite series (3.12) is absolutely convergent and bounded [5] so that the further manipulations of section 4 are valid; in other words the series (3.12) can be differentiated term-by-term to give the series (4.8) using the Rayleigh formula (4.5). As can be seen, there is ample room for classroom discussion on these points alone. While we are on this, point, the introduction of the double factorial $(2k-1)!!$ [7] (see, also, on this, Taser and Wang [18] with its definition and main identity (3.11) is a talking point in itself.

Finally, the derivation of the Rayleigh formula (4.5) in section 4 (Charlton [5]) gives an alternative derivation) provides an interesting exercise in the manipulation of linear differential operators, with the expression (4.1) being *resolved* into (4.3) by the use of the integrating factor (4.2). (The other relation in (2.9) and (2.10) yield similar differential exercises.) Equation (4.3), in turn, leads to the induction problem of proving (4.5) for arbitrary integers $m \geq 1$ moving on from this discussion of the basics of the paper itself, we note that we have restricted discussion to the *integral-ordered* Bessel equation (and functions)of the *first kind*. This leaves the possibility of generalizing the discussion to other types of Bessel equation and functions and to nonintegrable-order Bessel equation [13].

$$x^2\frac{d^2 i_n(x)}{dx^2}+x\frac{d i_n(x)}{dx}-(x^2+n^2)i_n(x)\equiv 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.4)$$

With modified Bessel function $i_(x)$ as particular solution and attempt to modify the present approach to tackle (5.4). In this case, a repetition of the analysis of section 2 shows that (5.4) can be re-expressed in the form (for example)

$$(\frac{d}{dx}-\frac{(n-1)}{x})(\frac{d}{dx}+\frac{n}{x})i_n(x)=i_n(x)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.5)$$

With the rest of the analysis following on as before. Or again, we could try to extend the integer-ordered (1.1) to an arbitrary-ordered $(n\longrightarrow v)$ Bessel equation

$$x^2\frac{d^2 j_v(x)}{dx^2}+x\frac{d j_v(x)}{dx}+(x^2-v^2)j_v(x)\equiv 0\ldots\ldots\ldots\ldots\ldots\ldots(5.5)$$

Then, on examining (4.8), it is tempting to simply replace the factorials by the obvious *gamma function*, $\Gamma(x)$, generalizations and write the solution to (5.5) as [13]

$$j_v(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(r+1)\Gamma(r+v+1)} \left(\frac{x}{2}\right)^{v+2r} \ldots\ldots\ldots\ldots\ldots\ldots (5.6)$$

Which is actually correct! (And, we have now introduced yet another special function). This last 'example' and the above series analysis can be used, then, to motivate the series solution of linear ordinary differential equation. Further, instead of adopting the hybrid approach developed above, it is possible to adopt a more direct attack on the problem through the application of the Laplace transform alone [5, 18]; the analysis of this paper is a handy introduction to the nuts and bolts of these approaches. In conclusion, a hybrid-approach to the solution of the Bessel equation and introduction of Bessel function has been presented. The approach develops the integral-order Bessel function through a combination of a factorization/ladder-operator technique with the Laplace transform which leads to the usual infinite series representation of the integral-order Bessel function as particular solutions of the integral-order Bessel equation. Along the way, various other notions involving differential operations, infinite series, Binomial coefficients, double factorials, etc., have been introduce and the possibility of further applications of the methods discussed.

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