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Common fixed point theorems for four mappings in fuzzy metric space

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Abstract
Our aim of this paper is to obtain a common fixed point theorem for four self mappings of generalized S-fuzzy metric space, which generalize the result of Singh and Chauhan. Many attempts have been made for proposing non additive models of uncertainty. Most radical attempt was initiated by L. Zadeh in 1965. Many authors have introduced the concept of fuzzy metric spaces in different ways. Kramosil and Michalek is one of them. Recently Singh and Chouhan developed a new concept of generalized fuzzy metric space (or s-fuzzy metric space) and proved Banach contraction principle in this newly developed space. In this paper we establish a general part of common fixed point theorem, which generalize the result of Singh and Chauhan.

Keywords: S-Fuzzy metric space, common fixed point, t-norm, R-weakly commuting pairs, reciprocally continuous maps

Introduction
Sessa [7] introduced a generalization of commutativity, so called weak commutativity. Further Jungck [12] introduced more generalized commutativity which is called compatibility in metric space and proved common fixed point theorems. Grabiec [3] proved fuzzy Banach contraction theorem on fuzzy metric space in the sense of [6]. Bijendra Singh and M.S. Chouhan [8] introduced the concept of compatibility in fuzzy metric space and proved some common fixed point theorems in fuzzy metric space in the sense of George and Veeramani [2]. Jungck and Rhoades [5] intruduced the notion of coincidentally commuting (or weakly compatible) mappings and obtained fixed point theorems for set-valued mapping. Also Dhage [1] introduced this concept and proved common fixed point theorems in D-metric space.
Recently, Bijendra Singh and Shishir Jain [9] intruduced the concept of weak compatibility in Menger space and proved common fixed point theorems in Menger space.

Preliminaries
1) Definition: The 3-tuple (X, S, *) is said to be a S-Fuzzy Metric Space if X is an arbitrary Set, * is a continuous t-norm and S is a Fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions.

i) $S(x, y, t) > 0$
ii) $S(x, y, t) = 1$ if and only if $x=y$
iii) $S(x, y, t) = S(y, x, t)$
iv) $S(x, y, t) * S(y, z, s) \leq S(x, z, t+s)$
v) $S(x, y, .); (0, \infty) \rightarrow [0, 1]$ is continuous for all $x, y, z \in X$ and $t, s > 0$

2) Definition: A sequence $\{x_n\}$ in a Fuzzy Metric Space (X, S, *) is a Cauchy Sequence if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that
$S(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$

3) Definition: A sequence $\{x_n\}$ in a Fuzzy Metric Space (X, S, *) is converges to x if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that
$S(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

4) Definition: Fuzzy Metric Space (X, S, *) is said to be complete if every Cauchy Sequence in (X, S, *) is a convergent sequence.

5) Definition: Two mappings f and g of a fuzzy metric space (X, S, *) in to itself are said to be weakly commuting if
$S(fgx, gfx, t) \geq S(fx, gx, t)$ for each x in X.

6) **Definition:** The mappings f and g of a fuzzy metric space (X, S, *) into itself are said to be R-weakly commuting, provided there exists some positive real numbers R such that
$S(fgx, gfx, t) \geq S(fx, gx, t/R)$ for each x in X.
7) **Definition:** The mappings F and G of a fuzzy metric space (X, S, *) in to itself are said to be compatible iff
$S(FGx_n, GFx_n, t) \to 1$
For all t > 0, whenever $\{x_n\}$ is a sequence in X such that
$Fx_n, Gx_n \to y$ for some y in X.
8) **Definition:** Let A and B be self mappings of a fuzzy metric space (X, S, *), we will call A and B to be reciprocally continuous if

$\lim_{n\to\infty} ABx_n = Ap$ and $\lim_{n\to\infty} BAx_n = Bp$
whenever $\{x_n\}$ is a sequence such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = p$ for some p in X

If A and B are continuous then they are obviously reciprocally continuous. But the converse need not be true.
The following theorem was proved by Balasubramaniam.

Theorem: Let (A, E) and (B, F) be point wise R-weakly commuting pairs of self mappings of complete fuzzy metric space (X, M, *) such that
a) $AX \subset FX$, $BX \subset EX$
b) $M(Ax, By, t) \geq M(x, y, ht)$, $0<h<1$; $x, y \in X$ and $t>0$

Suppose that (A, E) and (B, F) is compatible pairs of reciprocally continuous mappings. Then A, B, E and F have a unique common fixed point.
The following theorem was proved by Pant and Jha [7].

Theorem: Let (A, E) and (B, F) be pointwise R-weakly commuting pairs of self mappings of complete fuzzy metric space (X, M, *) such that
a) $AX \subset FX$, $BX \subset EX$
b) $M(Ax, By, t) \geq M(x, y, ht)$, $0<h<1$; $x, y \in X$ and $t>0$

Let (A, E) and (B, F) be compatible mappings. If any of the mappings in compatible pairs (A, E) and (B, F) is continuous then A, B, E and F have a unique common fixed point.
Singh B. and Chauhan M.S. [8] have proved the following theorem.

Theorem [8]: Let A, B, E and F be self maps of a complete fuzzy metric space (X, M, *) with continuous t-norm, * defined by $a*b = \min\{a, b\}$; $a, b \in [0, 1]$ satisfying the following conditions.
i) $A(x) \subset F(x)$, $B(x) \subset E(x)$.
ii) E and F are continuous.
iii) [A, E], [B, F] are compatible pairs of maps.
iv) For all x, y in X, $k \in (0, 1)$, $t>0$
$M(Ax, By, kt) \geq \min\{M(Ex, Fy, t), M(Ax, Ex, t), M(By, Fy, t), M(By, Ex, 2t), M(Ax, Fy, t)\}$
v) For all x, y in X, $\lim M(x, y, t) \to 1$ as $t \to \infty$

Then A, B, E and F have a unique common fixed point in X.
We shall establish the following theorems.
1) **Theorem:** Let A, B, M and N be self maps of a complete fuzzy metric space (X, S, *) with continuous t – norm * defined by
$a*b = \min\{a,b\}$, $a,b \in [0,1]$ satisfying the following conditions
$A(x) \subset N(x)$, $B(x) \subset M(x)$ …..(1.1)
[A, M], [B, N] are pointwise R-weakly commuting pairs of maps. …..(1.2)
[A, M] or [B, N] is compatible pair of reciprocally continuous maps. …..(1.3)
For all x, y in X, $k \in [0,1]$ $t>0$ …..(1.4)
$S^2(Ax, By, kt) \geq \max\{S^2(Mx, Ny, t), S^2(Ax, Mx, t), S^2(By, Ny, t), S(By, Ny, t), S(By, Mx, 2t)/2, S(Ax, Ny, 2t)/2\}$
1.1 For all x, y in X, $\lim S(x, y, t) \to 1$ as $t \to \infty$ …..(1.5)

Then A, B, M and N have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary. Construct a sequence $\{y_n\}$ such that

$y_{2n-1} = Nx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Mx_{2n} = Bx_{2n-1}$ n=1, 2, 3, Now using (1.4) we have

$S^2(y_{2n+1}, y_{2n+2}, kt) = S^2(Ax_{2n}, Bx_{2n+1}, kt)$

$\geq \max\{S^2(Mx_{2n}, Nx_{2n+1}, t), S^2(Ax_{2n}, Mx_{2n}, t), S^2(Bx_{2n+1}, Nx_{2n+1}, t), S(Bx_{2n+1}, Nx_{2n+1}, t), S(Bx_{2n+1}, Mx_{2n}, 2t)/2, S(Ax_{2n}, Nx_{2n+1}, 2t)/2\}$

$\geq \max\{S^2(y_{2n}, y_{2n+1}, t), S^2(y_{2n+1}, y_{2n}, t), S^2(y_{2n+2}, y_{2n+1}, t), S(y_{2n+2}, y_{2n+1}, t), S(y_{2n+2}, y_{2n}, 2t)/2, S(y_{2n+1}, y_{2n+1}, 2t)/2\}$

$\geq \max\{S^2(y_{2n}, y_{2n+1}, t), S^2(y_{2n+1}, y_{2n+2}, t), S(y_{2n+2}, y_{2n+1}, t), S(y_{2n+1}, y_{2n+1}, 2t)/2, S(y_{2n}, y_{2n+2}, 2t)/2\}$

$\Rightarrow S(y_{2n+1}, y_{2n+2}, kt) \geq S(y_{2n}, y_{2n+1}, t)$ …..(1.6)

Further using (1.4) we have

$S^2(y_{2n}, y_{2n+1}, kt) = S^2(Bx_{2n-1}, Ax_{2n}, kt)$

$= S^2(Ax_{2n}, Bx_{2n-1}, kt)$

$\geq \max\{S^2(Mx_{2n}, Nx_{2n-1}, t), S^2(Ax_{2n}, Mx_{2n}, t), S^2(Bx_{2n-1}, Nx_{2n-1}, t), S(Bx_{2n-1}, Nx_{2n-1}, t), S(Bx_{2n-1}, Mx_{2n}, 2t)/2, S(Ax_{2n}, Nx_{2n-1}, 2t)/2\}$

$\geq \max\{S^2(y_{2n}, y_{2n-1}, t), S^2(y_{2n+1}, y_{2n}, t), S^2(y_{2n}, y_{2n-1}, t), S(y_{2n}, y_{2n-1}, t), S(y_{2n}, y_{2n}, 2t)/2, S(y_{2n+1}, y_{2n-1}, 2t)/2\}$

$\Rightarrow S(y_{2n}, y_{2n+1}, kt) \geq S(y_{2n-1}, y_{2n}, t)$ …..(1.7)

Using (1.6) and (1.7) we have

$S(y_n, y_{n-1}, (1-k)t/k) \geq S(y_{n-1}, y_{n-2}, (1-k)t/k^2)$

$\geq S(y_{n-2}, y_{n-3}, (1-k)t/k^3)$

$\geq S(y_0, y_1, (1-k)t/k^n) \rightarrow 1$ as $n \rightarrow \infty$

Hence for $t > 0$, k, $\lambda \in (0, 1)$ we can choose $n_0 \in N$ such that

$S(y_n, y_{n-1}, (1-k)t/k) \geq 1- \lambda \quad n \geq n_0$ ……(1.8)

To prove that $\{y_n\}$ is a Cauchy Sequence, we claim (1.9) is true for all $n \geq n_0$ and for every $m \in N$

$S(y_n, y_{n+m}, t) \geq 1- \lambda$ ……(1.9)

From (1.6), (1.7) and (1.8) we have

$S(y_n, y_{n+1}, t) \geq S(y_n, y_{n-1}, t/k)$

$\geq S(y_n, y_{n-1}, (1-k)t/k)$

$\geq 1- \lambda$

Thus result (1.9) is true for m = 1. Further Suppose (1.9) is true for m.
Then we shall show that it is also true for m+1.
Using (1.6) (1.7) and definition for t – norm we have

$S(y_n, y_{n+m+1}, t) \geq S(y_{n-1}, y_{n+m}, t/k)$,

$\geq \min(S(y_n, y_{n-1}, (1-k)t/k), S(y_n, y_{n+m}, t)\}$

$\geq 1- \lambda$

Thus (1.9) is true for m+1 and so it is true for every m $\in$ N
Therefore $\{y_n\}$ is a Cauchy Sequence.
Since (X, S, *) is complete so $\{y_n\}$ converges to some point z in X. Thus $\{Ax_{2n}\}$ $\{Mx_{2n}\}$ $\{Bx_{2n-1}\}$ and $\{Nx_{2n-1}\}$ also converges to z.
Suppose [A, M] is a compatible pair of reciprocally continuous maps. Then by the definition of reciprocally continuous maps

$AMx_{2n} \rightarrow Az$ and $MAx_{2n} \rightarrow Mz$

And then the compatibility of A and M yields

$\lim_{n\to\infty} S(AMx_{2n}, MAx_{2n}, t) = 1$

i.e. S(Az, Mz, t) = 1

Hence Az = Mz

Since A(x) $\subset$ N(x), There exists a point w in X such that

Az = Nw

Using (1.4) we have
$S^2(Az, Bw, kt) \geq \max \{S^2(Mz, Nw, t), S^2(Az, Mz, t), S^2(Bw, Nw, t), S(Bw, Nw, t), S(Bw, Mz, 2t)/2, S(Az, Nw, 2t)/2\}$
$\geq \max \{S^2(Az, Az, t), 1, S^2(Bw, Az, t), S(Bw, Az, t), S(Bw, Az, 2t)/2, S(Az, Az, 2t)/2\}$

Or

$S^2(Az, Bw, kt) \geq 1$

Which implies that Az =Bw
Thus

Mz = Az = Nw = Bw.

Point-wise R-weakly commutativity of A and M implies that there exists R > 0 such that
$S(AMz, MAz, t) \geq S(Az, Mz, t/R) = 1$
i.e. AMz = MAz
and AAz = AMz = MAz = MMz

Similarly pointwise R-weakly commutativity of B and N implies that
BBw = BNw = NBw = NNw

Now by (1.4) we have
$S^2(AAz, Az, kt) = S^2(AAz, Bw, kt)$
$\geq \max \{S^2(MAz, Nw, t), S^2(AAz, MAz, t), S^2(Bw, Nw, t), S(Bw, Nw, t), S(Bw, MAz, 2t)/2, S(AAz, Nw, 2t)/2\}$
$\geq \max \{S^2(AAz, Az, t), 1, S^2(Az, Az, t), S(Az, Az, t), S(Az, AAz, 2t)/2, S(AAz, Az, 2t)/2\}$
Or
$S^2(AAz, Az, kt) \geq 1$
$\Rightarrow$ AAz = Az
Thus Az = AAz = MAz

Thus Az is a common fixed point of A and M
Again by (1.4) we have
$S^2(Az, BBw, kt) \geq \max \{S^2(Mz, NBw, t), S^2(Az, Mz, t), S^2(BBw, NBw, t), S(BBw, NBw, t), S(BBw, Mz, 2t)/2, S(Az, NBw, 2t)/2\}$
$\geq \max \{S^2(Az, BBw, t), 1, S(BBw, Az, 2t)/2, S(Az, BBw, 2t)/2\}$
Or $S^2(Az, BBw, kt) \geq 1$
$\Rightarrow$ Az = BBw

Thus
Az = BBw = Bw

Thus Bw(=Az) is a common fixed point of B and N and hence Az is a common fixed point of A, B, M and N.
To prove Uniqueness, let Az_1 be another common fixed point of A, B, M and N. Then we have

$S^2(Az, Az_1, kt) = S^2(AAz, BAz_1, kt)$
$\geq \max \{S^2(MAz, NAz_1, t), S^2(AAz, MAz, t), S^2(BAz_1, NAz_1, t), S(BAz_1, NAz_1, t), S(BAz_1, MAz, 2t)/2, S(AAz, NAz_1, 2t)/2\}$
$\geq \max \{S^2(Az, Az_1, t), S^2(Az, Az, t), S^2(Az_1, Az_1, t), S(Az_1, Az_1, t), S(Az_1, Az, 2t)/2, S(Az, Az_1, 2t)/2\}$
$\geq \max \{S^2(Az, Az_1, t), 1, 1,1, S(Az_1, Az, 2t)/2, S(Az, Az_1, 2t)/2\}$
or $S^2(Az, Az_1, kt) \geq 1$
$\Rightarrow Az = Az_1$

Thus, Az is a unique common fixed point of A, B, M and N.

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